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| Infinite seq | | Seq: a1, a2,..., an, ... . ai: general term of the seq  {an} to denote the infinite seq or e.g. {2n} or or (an)  Alternate sign an = (-1)n+1 or (-1)n  Recurrence eqn: e.g. Fibonacci seq (fn+2 = fn + fn+1) | | | | | | | | | Triangular Numbers: an = n(n+1)/2.  Hexagonal Numbers: an = n(2n-1)  Arithmetic seq: an = a + (n-1)d  Geometric seq: an = arn-1 | | | | | | | |
| Limits of seq | | L is the limit of {an} if for sufficiently large n, an is close to L where L is a finite num. i.e. = L or an L  If limit of {an} exists, it is unique. If = L, then {an} is convergent and {an} converges to L. If DNE, {an} is divergent  Precise defn: L is limit of (xn) if for every > 0, K s.t. |xn - L| < for all n ≥ K  If (xn) converges, then it has exactly 1 limit | | | | | | | | | | | | | | | | |
|  | | Ways to solve: simplify, divide by highest power of n in P and Q or multiply by conjugate | | | | | | | | | | | | | | | | |
| Standard Results | | = 1 for any a > 0 | | | | | | = 1 | | | = 0 for - 1 < r < 1 | | | | | = ea for any a | | |
| Let {an} and {bn} be 2 convergent seq w = A and = B  1. = cA and = c + A, c  2. = A ± B | | | | | | | | | | | 3. = AB  4. = , if B ≠ 0 and bn ≠ 0 for all n  5. = f(A) if f is a cts fn and limit of left exists | | | | | |
| If = 0, then = = 1 and = = 1  In particular when c = 0 and g(x) = x, = = 1 and = = 1 | | | | | | | | | | | | | | | | |
| Using fn | | Suppose there is a fn s.t. an = f(n). If = L, then = L (i.e. what is true for f(x) also true for an, might not be true the other way). f(x) is for x but an = f(n) is for n | | | | | | | | | | | | | | | | |
| L'Hopital Rule | | Let f and g be cts at x = a. Suppose f(a) = g(a) = 0. Then is of the form indeterminate form  L'Hopital rule: suppose f and g are differentiable in neighbourhood of c and f(c) = g(c) = 0 and g'(c) ≠ 0 except possibly at c, then = . form also works (sign of ∞ don't matter)  If f'(a) and g'(a) exists (≠ 0), then = | | | | | | | | | | | | | | | | |
|  | | | | 1) Solve = = ... = L. | | | | | | | | 2) Then = eL (ln ab = b ln a) | | | | | | |
| Squeeze theorem | | | | | If an ≤ bn ≤ cn for all n > n0. If = L = , then = L | | | | | | | | | | = 0 (prove using squeeze theorem) | | | |
| Monotonic seq | | | {an} is increasing if an ≤ an+1 for all n (show ≥ 1 or an+1 - an > 0) OR decreasing if an ≥ an+1 for all n  Both are monotonic seq (non-monotonic e.g. (-1)n)  C is lower bound of {an}if an ≥ C for all n and C is upper bound if an ≤ C for all n  {an} is bounded if it has an upper and lower bound | | | | | | | | | | | | | | | |
| Monotone Convergent Theorem | | | | | | | Increasing (decreasing) seq w upper (lower) bound is convergent | | | | | | | | | | | |
| Principle of Mathematical Induction (PMI) | | | | | | For ea n , let P(n) be statement about n. Show P(1) is true. Assume P(k) true, prove P(k+1) true. Then P(n) true n | | | | | | | | | | | | |
| Trigo | sin2 + cos2 = 1  1 + tan2 = sec2  1 + cot2 = csc2 | | | | | | | | sin(A ± B) = sinA cosB ± cosA sinB  cos(A ± B) = cosA cosB sinA sinB  tan(A ± B) = | | | | | sin (-x) = - sin x  cos(-x) = cos(x)  tan(-x) = - tan x | | | | csc(-x) = - csc(x)  sec(-x) = sec(x)  cot(-x) = - cot(x) |
| 1 - x2 ≤ cos x for -π/2 < x < π/2  x < tan x for 0 < x < π/2 | | | | | | | | | If a,b,c are sides of triangle and is angle opp c, then c2 = a2 + b2 -2abcos | | | | | | | | -|| ≤ sin ≤ ||  -|| ≤ 1 - cos ≤ || |
| sin 2 = 2sincos =  cos 2 = cos2 - sin2 =  tan 2 = | | | | | | | | | csc 2 =  sec 2 =  cot 2 = | | | | cos2 =  sin2 = | | | | = 1  = 0  = 1 |
| sin(π - x) = sin x  cos(π - x) = -cos x  tan(π - x) = -tan x | | | | | | | | | (a+b)3 = a3 + 3a2b + 3ab2 + b3  (a-b)3 = a3 - 3a2b + 3ab2 - b3  y-x = (y1/n-x1/n)(y(n-1)/n + y(n-2)/nx1/n +...+ x(n-1)/n) | | | | | | | =  = | |

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| Infinite Series | Infinite sequence {an}: a1,a2,...,an,...  Infinite Series: a1 + a2 + ... + an + ... = . an: nth term of the series. = is convergent iff exists | | | | | | |
| Partial Sum | | s1 = a1  s2 = a1 + a2  ...  sn = a1 + a2 + ... + an  ...  s1, s2,...,sn,... is the sequence of partial sums | | | sn is the nth partial sum =  = . So if = L, then = L  If sn L, then is convergent. If seq {sn} does not converge, then series is divergent  {sn} does not coverge: either DNE OR = ±∞  OR equivalently DNE OR = ±∞ | | |
| 1 - 1 + 1 - 1 + 1 - 1...  s1 = 1. s2 = 0. s3 = 1. s4 = 0 | | | Seq alternates btw 0 and 1, so DNE  Thus infinite sum is divergent | | |
| Convergence of Geometric Series | | For a ≠ 0, series a + ar + ar2 + ... + arn-1 = : geometric series, where a is the 1st term, r is the common ratio  nth partial sum sn = , r ≠ 1  |r| < 1: sn . |r| > 1: series diverges. |r| = 1: could converge or diverge | | | | | |
| Telescoping Series | | = + + + .... Use partial fraction to write = –  sn = + + + ... + = ( – ) + ( – ) + ( – ) + ... + ( – ) = 1 – . = = 1 | | | | | |
| Theorems | | If and are convergent, then is also convergent and = +  If is convergent and c , then is also convergent and = c | | | | | |
| nth Term Test | | Test for divergence only: If does not converge to 0, then is divergent (intuitively, an 0)  Inverse error: {an} converges to 0 does NOT necessarily imply converges  Proof: Let = S. S = . S = as well. Then an = sn - sn-1. Taking limits on both sides, = S - S = 0 | | | | | |
| p-series Test | p-series: is convergent if p > 1 and divergent if p ≤ 1 | | | | | Harmonic series: is divergent (p=1) | |
| Proof: sum of areas of rectangles over 1 ≤ x ≤ n  < = =  Let N ∞, then < (since N1-p 0 as (1-p) < 0), which is a finite value  Thus converges (starting with n=2 or n=1 does not matter) | | | | | | Proof: sum of areas of rectangles over 1 ≤ x ≤ n  > = =  Let N ∞, then RHS ∞ as (1-p) > 0  Thus diverges |
| Non-negative series | | | is an (eventually) non-negative series if each ak ≥ 0 (eventually) i.e. K s.t. ak ≥ 0 k ≥ K | | | | |
| Comparison Test | | Let and be 2 non-negative series. Suppose K s.t 0 ≤ ak ≤ bk k ≥ K  1. If converges, then converges. 2. If diverges, then diverges | | | | | |
| Limit Comparison Test | | Let and be 2 eventually positive series, and suppose limit p = exists  1. If p > 0, then either both series converge or both series diverge  2. If p = 0 and converges then converges (intuitively is smaller than) | | | | | |
| Ratio Test | | Let be an eventually positive series, and suppose limit p = exists. | | | | | |
| Root Test | | Let be an eventually non-negative series, and suppose limit p = exists. | | | | | |
| Alternating Series Test | | Alternating series is of the from: OR , w all ak ≥ 0  Alternating series Test: Suppose 1. an ≥ 0 n. 2. {an} is decreasing (i.e. an ≥ an+1 n). 3. = 0  Then OR is convergent | | | | | |
| Absolute Convergent / Conditionally Convergent / Divergent | | | | 1. converges absolutely if converges  2. converges conditionally if (i) converges, and (ii) diverges  If converges absolutely, then it converges  Every series is either absolutely convergent, conditionally convergent or divergent | | | |
| Remarks | | 1. nth term test for divergence ONLY, and no conclusion if 0  2. Comparison/Limit Comparison Test (use if given series looks like geometric/p-series)  - Usually easier to use LCT. But if given series has oscillating term, then try CT first  3. Ratio Test: use if given series looks like geometric series, contains n! or is defined recursively  4. Root Test: use if given series has high power such as nth power  5. Alternating Series Test: ony for alternating series | | | | | |

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| Power Series | Power Series about x = 0 is of the form = c0 + c1x + c2x2 + ... + cnxn + ... where ci, i=0,...,n are constants and x is var | | | | |
| Geometric series: a + ar + ar2 + ... + arn-1 + ... converges to the sum if |r| < 1 and diverges if |r| ≥ 1  = 1 + x + x2 + ... + xn + ... is geometric series where a = 1, r = x.  So this power series about x = 0 converges to when |x| < 1  = 1 + x + x2 + ... + xn + ... = , -1 < x < 1 aka Binomial expansion | | | | |
| Power series about x = a is = c0 + c1(x-a) + c2(x-a)2 + ... + cn(x-a)n + ... . a is the centre of the power series | | | | |
| Convergence & Radius of Convergence (R) of power series | | Power series always behave in 1 of the 3 ways: (i) Converges x; R = ∞  (ii) Converges only at x = a and diverges elsewhere (sub x = a, then can show converges to c0); R = 0  (iii) Converges x in interval (a-h, a+h) but diverges for x < a-h and x > a+h; R = h = [(a+h)-(a-h)]/2  Power series cannot be convergent for ≥ 2 disjoint intervals | | | |
| Ratio test: Let be a series and let = p. | | | |
| Differentia-tion and Integration of Power Series | | If has R = h, it defines a function f: f(x) = , a – h < x < a + h  Function f has derivatives of all orders in (a-h, a+h)  f'(x) = , f''(x) = ... Differentiates series also converges for a-h < x < a+h | | | |
| = + C, which also converges for a-h < x < a+h. CHECK for C  = 1 – t + t2 – ... ..., -1 < t < 1 from Binomial expansion (x = –t). Integrate both sides, ln(1+x) = x - + - ..., -1 < t < 1 | | | |
| Taylor Series | | Let f be a fn w derivatives of all orders throughout some interval containing a as an interior point  Taylor series of f at a: = f(a) + f'(a)(x-a) + ... + (x-a)n + ...  Use Taylor series to expand a fn into power series, just change ci to f(i)(a)/i!  Maclaurin Series (Taylor series of f at 0): f(x) = = f(0) + f'(0)x + ... + xn + ... | | | |
| Standard Taylor Series | | ex = 1 + x + + ... = , -∞ < x < ∞  sin x = x - + - ... = , -∞ < x < ∞  cos x = 1 - + - ... = (just sin x), -∞ < x < ∞  = 1 + (x-a) + (x-a)2 + ... + (x-a)n + ... = | | (sin-1 x) = , x ≠ ±1  (cos-1 x) = , x ≠ ±1  (sec-1 x) = , x ≠ ±1, 0  (csc-1 x) = , x ≠ ±1, 0 | (tan-1 x) = ,  (cot-1 x) = |
| ln(1+x) = x - + - ..., -1 < x < 1 | tan-1x = x - + - ..., -1 ≤ x ≤ 1 |
| Geometric Series | | - If need find Taylor series of f at x = a, convert f to have (x-a) somewhere in f  - If expanding Taylor series too difficult, use geometric series w the converted f | | | |
| Taylor Polynomials | | n-th order Taylor polynomial of f at a is Pn(x) = = f(a) + f'(a)(x-a) + ... + (x-a)n, which provides the best polynomial approximation of deg n | | | |

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| Cartesian Coordinate Sys | | Right hand rule: index/x, middle/y, thumb/z  3 planes divide 3D space into 8 octants  Dist btw 2 pts = |P1P2| = | | Eqn of circle: (x-x0)2 + (y-y0)2 = r2  Eqn of sphere: (x-x0)2 + (y-y0)2 + (z-z0)2 = r2 | | | | | |
| Vector | | Position vector of P(x0, y0, z0) = =  Length/magnitude of = = | | =  If c > 0, then c**v**1 and **v**1 point in same dirn, else opp dirn | | | | | |
| Angle btw 2 vectors | | = (x1-x2)2 + (y1-y2)2 + (z1-z2)2  Cosine rule: | | | | Define **v**1 **v**2 = x1x2 + y1y2 + z1z2  Then (0 ≤ ≤ 180˚)  perpendicular to iff | | | |
| Scalar/Dot Product | | **v**1 **v**1 = ≥ 0 | | = | | | | | |
| Unit vector | | Standard unit vector (length = 1): **i** = , **j** = , **k** = | So every vector can be written as x**i** + y**j** + z**k** | | | | | | Unit vector: |
| Projection | | Projection of **b** onto **a** (proj**ab**)  Note = = . So = | | Then proj**ab** = \* unit vector along a = = | | | | | |
| Vector Product | | Note = 0 =  = area of parallelogram form by and  // (parallel) iff | | | | |  | | |
| Lines | | Vector eqn of line: **r** = (x0**i** + y0**j** + z0**k**) + t(a**i** + b**j** + c**k**) OR **r**(t) = **r**0 + t**v**  Parametric eqn of line: | | | To find intersect, just equate both equation and compare **i,j,k** components. If simultaneous eqn match -> intersect  Don't match -> skew lines (lines on diff planes; don't intersect) | | | | |
| Shortest dist from pt to line | | Dist from to line where perpendicular to  Find and . Then = | | | OR AQ = | | | | |
| Planes | Vector Eqn of plane: (where is a pt on plane, is normal)  Cartesian eqn: ax + by + cz = d (where a,b,c are normal coordinates) | | | | | | | Dist from pt to plane =  dist btw 2 parallel plane = (same **unit** normal) | |
| Vector fns of 1 variable | | Consider vector fn **r**(t) = f(t)**i** + g(t)**j** + h(t)**k**  Parametric eqn: x = f(t), y = g(t), z = h(t)  The derivative of **r**(t) = **r**'(t) = = = tangent vector | | If f,g,h are differentiable fns, then **r**'(t) = f'(t)**i** + g'(t)**j** + h'(t)**k** | | | | | |
| Space Curves | | Plot point of vector fn **r**(t) for diff values of t | | **r**(t) = (r cos t)**i** + (r sin t)**j** OR x = r cos t, y = r sin t : circle | | | | | |
| Curve traced by **r** is smooth if (i) **r**'(t) is cts and (ii) **r**'(t) ≠ **0** for any values of t | | | | | | | |
| Unit tangent vector at t = t0 = ( ≠ **0**) | Arc length L = = | | | | | | |

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| Function in 2 variables | | Domain of f = Df = {(x,y)|f(x,y) is defined}  x2 + y2 = r2 (circle w center (0,0) w radius r) | | | (ellipse w center (0,0) w horizontal a, vertical b) | |
| Geometric Meaning | y = f(x) – curve in xy-plane  z = f(x,y) – surface in 3D space | | | To visualise: fix value of z, then we get a plane parallel to xy-plane. Imagine plane intersect with surface  Fix value of y, then we get a plane parallel to xz-plane, then imagine plane intersect with surface | | |
| Partial Derivatives | | Given f(x,y), Partial derivative of f w.r.t x at (a,b) = if limit exists (treat y as constant)  Partial derivative of f w.r.t y at (a,b) = if limit exists (treat x as constant) | | | | |
| Higher order partial derivatives | | 2nd order partial derivatives of f are: fxx = (fx)x = , fyy = (fy)y = , fxy = (fx)y = , fyx = (fy)x = ,  Note fxy (do from left to right); (do from right to left)  Let f(x,y) be a fn defined on a region D containing (a,b). If fx, fy, fxy, fyx are all cts in D, then fxy(a,b) = fyx(a,b)  For fn in 3 variables f(x,y,z), fx = , fy = , fz = | | | | |
| Chain Rule | | If z = f(x,y), and x = x(t), y = y(t), then z = f(x(t), y(t)). Can now find OR  If w = f(x,y,z) and x = x(t), y = y(t) and z = z(t), then  If z = f(x,y) and x = x(s,t), y = y(s,t), then  If w = f(x,y,z) and x = x(s,t), y = y(s,t) and z = z(s,t), then | | | | |
| Directional Derivatives | | is rate of change of f w.r.t x (along dirn of x-axis) at (a,b)  is rate of change of f w.r.t y (along dirn of y-axis) at (a,b)  Direction derivative f at (a,b) in dirn of **unit** vector **u** = u1**i** + u2**j** is if limit exists  Note D**i**f(a,b) = fx(a,b) and D**j**f(a,b) = fy(a,b) | | | | |
| Physical Meaning | | | measures change in value df of fn f when we move a small dist dt from pt (a,b) in dirn of **u**: | | | |
| Functions of 3 variables | | if limit exists =  for unit vector **u** = u1**i** + u2**j** + u3**k** since | | | | |
| Max and min values | | f(x,y) has local maximum at (a,b) if f(x,y) ≤ f(a,b) for all points (x,y) near (a,b). f(a,b) is a local maximum value  f(x,y) has local minimum at (a,b) if f(x,y) ≥ f(a,b) for all points (x,y) near (a,b). f(a,b) is a local minimum value | | | | |
| Critical points | | - A point (a,b) is called a critical point of f if (i) fx(a,b) = 0 and fy(a,b) = 0; or (ii) fx(a,b) or fy(a,b) DNE  - Let (a,b) be a point of f w fx(a,b) = 0 and fy(a,b) = 0. (a,b) is a saddle point of f is there are some dirn along which f has a local max at (a,b) and some dirn along which f also has a local min at (a,b)  - f may have a local max or min at (a,b) where fx(a,b) or fy(a,b) DNE | | | | |
| 2nd derivative test | | Assume f and its first and second partial derivatives are cts in a region containing (a,b) s.t. fx(a,b) = 0 and fy(a,b) = 0  Let D = fxx(a,b)fyy(a,b) – fxy(a,b)2   |  |  |  |  |  | | --- | --- | --- | --- | --- | | D | > 0 | > 0 | < 0 | = 0 | | fxx(a,b) OR fyy(a,b) | > 0 | < 0 |  |  | | (a,b) | local min | local max | saddle point | no conclusion | | | | | |
| Lagrange Multipliers | | Find relative extrema of z = f(x,y) subject to the constraint of g(x,y)  Construct fn F(x,y,) = f(x,y) – g(x,y). Then set Fx = 0, Fy = 0, = 0 and solve simultaneous eqn.  If w = f(x,y,z) subject to contraint of g(x,y,z)  Let F(x,y,z,) = f(x,y,z) – g(x,y,z). Then set Fx = 0, Fy = 0, Fz = 0, = 0 and solve simultaneous eqn. | | | | Note = 0 is trivial case |

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| Double Integrals | | Area under curve: A = =  Vol under surface over region R = , where R is a 2D region  Can treat f(x,y) sort of like the height | | | | | | Riemann integrals: Divide [a,b] into n equal intervals.  Length of each interval =  Riemann sum/Area under curve of y = f(x) from a to b ≈  A = = | | |
| Properties of Double Integrals | | , where c is a constant  = area of R  If R = R1 R2 and R1, R2 do not overlap except perhaps on boundary, | | | | | | | If f(x,y) ≥ g(x,y) for all points (x,y), then  If m ≤ f(x,y) ≤ M for all pts (x,y) in R, then | |
| Rectangular Regions | | R is a rectangular region bounded by: a ≤ x ≤ b and c ≤ y ≤ d, then  If f(x,y) = g(x)h(y), then | | | | | | | | |
| Type A region | | | Vertical line meets top and bottom boundaries. R: g1(x) ≤ y ≤ g2(x), a ≤ x ≤ b. Then | | | | | | | |
| Type B region | | | Horizontal line meets left and right boundaries. R: h1(y) ≤ x ≤ h2(y), c ≤ y ≤ d. Then | | | | | | | |
| Changing order of integration | | Sketch out region first | | | | | | | | |
| = | | | | |  | | | = |
| Double Integrals in Polar Coordinates | | x = r cos , y = r sin , x2 + y2 = r2  If R: a ≤ r ≤ b, , then | | | | Unit Circle: R: 0 ≤ r ≤ 1, 0 ≤ ≤ 2π  Ring: R: 1 ≤ r ≤ 2, 0 ≤ ≤ 2π | | | | Sector: R: 0 ≤ r ≤ 1, 0 ≤ ≤  Polar Rectangular:  R: a ≤ r ≤ b, |
| Integration by parts | | | |  | |  |  |  | | --- | --- | --- | | |x| < a |  | |x| > a | | |x| < a |  | |x| > a | | | | | | |
| Surface Area | S = | | | |
| Triple Integral | | Let D be the solid region in xyz space. Subdivide D into smaller cubic region Di (i=1,....,n). Let Vi be vol of V1 and (xi, yi, zi) be a pt in Di.  Then  Only consider rectangular region. D: a ≤ x ≤ b, c ≤ y ≤ d, r ≤ z ≤ s. Then | | | | | | | | |

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| Mass and Center of Gravity | If a lamina w cts density fn (x,y) occupies a region R in the xy-plane, its total mass M = , and its center of gravity () is , |
| Note if (x,y) is a constant, then the center of gravity is , |